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裂纹面受两对集中剪力作用下的弹塑性分析

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摘 要: 利用裂纹线场方法对理想弹塑性材料无限大板受两对集中剪力问题进行了弹塑性分析, 并且获得了理论解。这个解包括: 裂纹线附近弹塑性边界上的单位法向矢量、裂纹线附近的弹塑性解析解、裂纹线上的塑性区长度随荷载的变化规律及其承载力。分析不受小范围屈服假设的限制, 并且不附加假使条件。结果在裂纹线附近足够精确。

关键词: 裂纹线场方法; 两对集中剪力; 弹塑性解; 塑性区长度; 承载力

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ELASTOPLASTIC ANALYSIS OF AN INFINITE PLATE WITH A CRACK LOADED BY TWO PAIRS OF CONCENTRATED SHEAR FORCES

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Abstract: The near crack line analysis method is used to investigate a crack loaded by two pairs of point shear forces in an elastic-perfectly plastic infinite plate. The analytical solutions are obtained. The solutions include: the unit normal vector of the elastic-plastic boundary near the crack line, the elastic-plastic stress fields near the crack line, the variation law of the length of the plastic zone along the crack line with external loads, and the bearing capacity. The results are sufficiently precise near the crack line since the assumption of the small scale yielding theory is not used and no other assumptions are made.

Key words: near crack line analysis method; two pairs of concentrated shear forces; elastoplastic analytical solution; length of the plastic zone; bearing capacity

预测和估计无限大板中裂纹面受两集中剪力作用下的断裂行为是许多研究者感兴趣的问题。由于该问题的复杂性, 这个问题需要进一步解决。为了研究无限大板中裂纹面受两集中剪力作用下的断裂行为, 研究者们提出和发展了许多方法。在这些方法中, 裂纹线场方法被证明是有效的。裂纹线场方法首先由 Achenbach 和 Li^[1]提出。Achenbach 和 Dunayevsky^[2]以及 Guo 和 Li^[3]利用裂纹线场方法对理想弹塑性 型准静态扩展裂纹线附近的应力

和变形场进行了进一步研究。但是他们的研究是在小范围屈服理论的假设基础上得到的。小范围屈服理论的假使: 裂纹前缘塑性区很小, 塑性区外的弹性场为裂纹的线弹性奇异 K 场, 且塑性区的存在使裂纹的弹性场沿裂纹线向前移动了一微小距离。因此他们的结果对于塑性区不断增大的情况不适用。基于 Achenbach 和 Dunayevsky^[2]以及 Guo 和 Li^[3]工作, Yi^[4,5]对线场分析方法进行了改进, 使之不受小范围屈服理论的限制。为此, 吴承平和王成^[6]对

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裂纹面任意点受反平面集中力时进行了裂纹线场的弹塑性分析。Zhou 和 Wang^[7]对裂纹面受一对集中剪力作用时进行了裂纹线场的弹塑性分析。Wang 和 Zhou^[8]利用改进的裂纹线场理论研究了在无限大板中裂纹面受两集中拉力作用下的断裂行为。Zhou 和 Wang^[9]利用改进的裂纹线场理论研究了岩体发生劈裂破坏的机理。周小平等^[10]研究了在一对集中拉力作用下一偏心裂纹的弹塑性问题。实际工程中往往存在裂纹面受两集中剪力作用的问题。为此,分析裂纹面受两集中剪力作用下的弹塑性问题具有重要的理论和现实意义。

1 理论分析

1.1 基本方程

对图 1 所示的平面问题,不为零的应力分量为 $\sigma_x, \sigma_y, \tau_{xy}$ 。平衡微分方程为:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad (1)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \quad (2)$$

假设材料满足 Mises 准则:

$$\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2 = 3k^2 \quad (3)$$

式中: k 为剪切屈服极限。

1.2 裂纹线附近的塑性应力场的通解

在裂纹线附近的塑性区的应力场按 y 的幂级数展开,并略去 y^3 以上的项数:

$$\sigma_x = f_1(x)y + o(y^3) \quad (4)$$

$$\sigma_y = m_1(x)y + o(y^3) \quad (5)$$

$$\tau_{xy} = s_0(x) + s_2(x)y^2 + o(y^4) \quad (6)$$

将式(4)~式(6)代入式(1)、式(2)和式(3),比较等式两端 y 的同次幂的系数的一组方程,解这组方程得 $f_1(x), m_1(x), s_0(x), s_2(x)$, 然后代入式(4)、式(5)和式(6)得裂纹线附近的塑性应力场:

$$\sigma_x = -3k \frac{y}{x+B} + o(y^3) \quad (7)$$

$$\sigma_y = 0 \quad (8)$$

$$\tau_{xy} = k - 1.5k \frac{y^2}{(x+B)^2} + o(y^4) \quad (9)$$

式中: B 为积分常数,由弹塑性边界上的匹配条件确定。式(7)、式(8)和式(9)是裂纹线塑性区应力场上的通解。

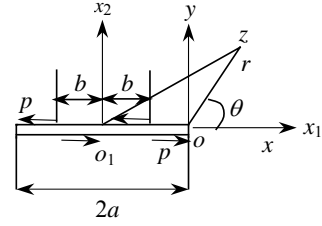


图 1 无限板中心裂纹受两对集中剪力作用

Fig.1 A centric crack loaded by two pairs of point shear forces in an infinite body

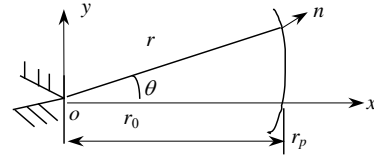


图 2 裂纹线附近的弹塑性边界

Fig.2 The elastic-plastic boundary

1.3 裂纹线附近的弹性应力场

对图 1 所示的裂纹问题,采用 Westergaard 应力函数:

$$Z_{II}(z) = \frac{2pz\sqrt{a^2 - b^2}}{\pi(z^2 - b^2)\sqrt{z^2 - a^2}} \quad (10)$$

则应力分量为:

$$\sigma_x = 2 \operatorname{Im} Z_{II}(z) + y \operatorname{Re} Z'_{II}(z) \quad (11)$$

$$\sigma_y = -y \operatorname{Re} Z'_{II}(z) \quad (12)$$

$$\tau_{xy} = \operatorname{Re} Z_{II}(z) - y \operatorname{Im} Z'_{II}(z) \quad (13)$$

式中:

$$Z'_{II}(z) = \frac{dZ_{II}(z)}{dz}, \quad z = x_1 + ix_2 (i = \sqrt{-1})$$

根据 Westergaard 应力函数,其应力强度因子为:

$$K_{II} = \lim_{|\xi| \rightarrow 0} \frac{2p(\xi+a)\sqrt{a^2 - b^2}\sqrt{2\pi\xi}}{\pi[(\xi+a)^2 - b^2]\sqrt{\xi(\xi+2a)}} = \frac{2p\sqrt{a}}{\sqrt{\pi(a^2 - b^2)}} \quad (14)$$

式中: $\xi = z - a$

将应力分量按小 θ 展开,并略去 θ^3 以上的项可得:

$$\sigma_x = -\frac{6\theta pr\sqrt{a^2 - b^2}}{\pi[r(2a+r)]^{3/2}[(a-b+r)(a+b+r)]^2} \times [a^4 + 6a^3r + 8ar^3 + 2r^4 - a^2(b^2 - 11r^2)] \quad (15)$$

$$\sigma_y = \frac{2pr\theta\sqrt{a^2 - b^2}}{\pi[r(2a+r)]^{3/2}[(a-b+r)(a+b+r)]^2} \times [a^4 + 6a^3r + 8ar^3 + 2r^4 - a^2(b^2 - 11r^2)] \quad (16)$$

$$\tau_{xy} = \frac{2p(a+r)\sqrt{a^2-b^2}}{\pi[r(2a+r)]^{1/2}[(a+r)^2-b^2]} - \frac{p\sqrt{a^2-b^2}\{5a^7+40a^6r+8a^7+a^5(138r^2-10b^2)+a^4(-44b^2r+258r^3)+a^3(5b^4-54b^2r^2+279r^4)+a^2(4b^4r-22b^2r^3+174r^5)+a(-2b^2r^4+58r^6)\}\theta^2/\{\pi[r(2a+r)]^{1/2}(a-b+r)^3(a+b+r)^3\}}{(17)}$$

式(15)、式(16)和式(17)为精确解,且满足裂纹线附近的所有边界条件。

如图2所示,弹塑性边界定义为 $r=r_p(\theta)$,根据 $r_p(\theta)$ 关于 θ 的对称性,按小 θ 展开得(略去 θ^3 以上的项):

$$r_p(\theta) = r_0 + r_2\theta^2 \quad (18)$$

式中: r_0 为零次方项裂纹线上的塑性区的长度, r_2 为二次方项塑性区的长度。

根据式(18),可得弹塑性边界上的单位法向量 $\mathbf{n}=(n_x, n_y)$ 为:

$$\left. \begin{aligned} n_x &= 1 - 0.5B_1\theta^2 \\ n_y &= B_1\theta \end{aligned} \right\} \quad (19)$$

式中: $B_1 = 1 - 2r_2/r_0$

在弹塑性边界上,根据式(19)有:

$$\left. \begin{aligned} x &= r_0 + r_0\left(\frac{r_2}{r_0} - \frac{1}{2}\right)\theta^2 \\ y &= r_0\theta \end{aligned} \right\} \quad (20)$$

将式(20)代入式(7)~式(9)得弹塑性边界上的塑性区应力场的表达式:

$$\sigma_x = -3k \frac{r}{x+B} \quad (21)$$

$$\sigma_y = 0 \quad (22)$$

$$\tau_{xy} = k - 1.5k \frac{r^2}{(x+B)^2} \theta^2 \quad (23)$$

将式(18)代入式(15)~式(17)得弹塑性边界上的弹性区应力场的表达式:

$$\begin{aligned} -3kB_1 \frac{r_0}{r_0+B} - 1.5k \frac{r_0^2}{(r_0+B)^2} - 2B_1^2k &= \frac{2p(-2B_1^2+1)(a+r_0)\sqrt{a^2-b^2}}{\pi[r_0(2a+r_0)]^{1/2}[(a+r_0)^2-b^2]} - p\sqrt{a^2-b^2}\{5a^7+40a^6r_0+8a^7+ \\ &a^5(138r_0^2-10b^2)+a^4(-44b^2r_0+258r_0^3)+a^3(5b^4-54b^2r_0^2+279r_0^4)+a^2(4b^4r_0-22b^2r_0^3+174r_0^5)+ \\ &a(-2b^2r_0^4+58r_0^6)\}/\{\pi[r_0(2a+r_0)]^{1/2}(a-b+r_0)^3(a+b+r_0)^3\} - \frac{p(7B_1r_0+1)\sqrt{a^2-b^2}}{\pi[r_0(2a+r_0)]^{3/2}(a-b+r_0)^2(a+b+r_0)^2} \times \end{aligned}$$

$$\sigma_x = -\frac{6pr_0\theta\sqrt{a^2-b^2}}{\pi[r_0(2a+r_0)]^{3/2}(a-b+r_0)^2(a+b+r_0)^2} [a^4+6a^3r_0+8ar_0^3+2r_0^4-a^2(b^2-11r_0^2)] \quad (24)$$

$$\sigma_y = \frac{2pr_0\theta\sqrt{a^2-b^2}}{\pi[r_0(2a+r_0)]^{3/2}(a-b+r_0)^2(a+b+r_0)^2} [a^4+6a^3r_0+8ar_0^3+2r_0^4-a^2(b^2-11r_0^2)] \quad (25)$$

$$\begin{aligned} \tau_{xy} &= \frac{2p(a+r_0)\sqrt{a^2-b^2}}{\pi[r_0(2a+r_0)]^{1/2}[(a+r_0)^2-b^2]} - \frac{p\sqrt{a^2-b^2}\{5a^7+40a^6r_0+8a^7+a^5(138r_0^2-10b^2)+a^4(-44b^2r_0+258r_0^3)+a^3(5b^4-54b^2r_0^2+279r_0^4)+a^2(4b^4r_0-22b^2r_0^3+174r_0^5)+a(-2b^2r_0^4+58r_0^6)\}\theta^2/\{\pi[r_0(2a+r_0)]^{1/2}(a-b+r_0)^3(a+b+r_0)^3\} - \\ &\frac{2pr_2\sqrt{a^2-b^2}\theta^2}{\pi[r_0(2a+r_0)]^{3/2}(a-b+r_0)^2(a+b+r_0)^2} \times [a(a^3+6a^2r_0+8r_0^3)+2r_0^4-a^2(b^2-11r_0^2)] \quad (26) \end{aligned}$$

在弹塑性边界上应力的法向分量和切向分量

分别为:

$$\left. \begin{aligned} \sigma_{nn} &= \sigma_x n_x^2 + \sigma_y n_y^2 + 2\tau_{xy} n_x n_y \\ \sigma_{ns} &= (n_x^2 - n_y^2)\tau_{xy} + (\sigma_y - \sigma_x)n_x n_y \end{aligned} \right\} \quad (27)$$

将式(21)~式(23)和式(19)代入式(27)可得弹塑性边界上靠近塑性区一侧的塑性区应力分量 $\sigma_{nn}^p, \sigma_{ns}^p$;将式(24)~式(26)和式(19)代入式(27)可得弹塑性边界上靠近弹性区一侧的弹性区应力分量 $\sigma_{nn}^e, \sigma_{ns}^e$ 。根据匹配条件,即在弹性区和塑性区边界上的应力必须满足连续($\sigma_{nn}^p = \sigma_{nn}^e, \sigma_{ns}^p = \sigma_{ns}^e$),然后比较等式两端的同次幂的系数可得下列方程:

$$k = \frac{2(p-kr_0)(a+r_0)\sqrt{a^2-b^2}}{\pi[r_0(2a+r_0)]^{1/2}[(a+r_0)^2-b^2]} \quad (28)$$

$$\begin{aligned} 2B_1k &= -\frac{6pr_0\sqrt{a^2-b^2}}{\pi[r_0(2a+r_0)]^{3/2}(a-b+r_0)^2(a+b+r_0)^2} \times \\ &[a^4+6a^3r_0+8ar_0^3+2r_0^4-a^2(b^2-11r_0^2)] + \\ &4B_1 \frac{p(a+r_0)\sqrt{a^2-b^2}}{\pi[r_0(2a+r_0)]^{1/2}[(a+r_0)^2-b^2]} + \frac{3kr_0}{r_0+B} \quad (29) \end{aligned}$$

$$[a^4 + 6a^3r_0 + 8ar_0^3 + 2r_0^4 - a^2(b^2 - 11r_0^2)] \quad (30)$$

根据式(28)可以获得裂纹线附近的塑性区长度,然后将式(28)代入式(29)和式(30)可以获得 B 和 B_1 。

通过引用无量纲参数 $\xi = r_0/a$, $\lambda = b/a$, $\eta = 2p/(2ak)$, 式(28)可以写成下列形式:

$$\eta = \xi + \frac{\pi(2\xi + \xi^2)^{1/2}[(1 + \xi)^2 - \lambda^2]}{2(1 + \xi)\sqrt{1 - \lambda^2}} \quad (31)$$

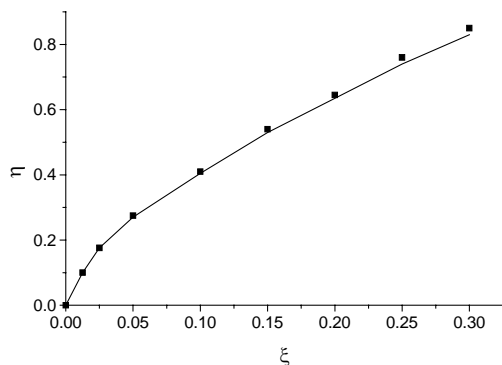


图 3 当 $\lambda = 0.1$ 时 η 随 ξ 的变化

Fig.3 Dependence of η on ξ when $\lambda = 0.1$

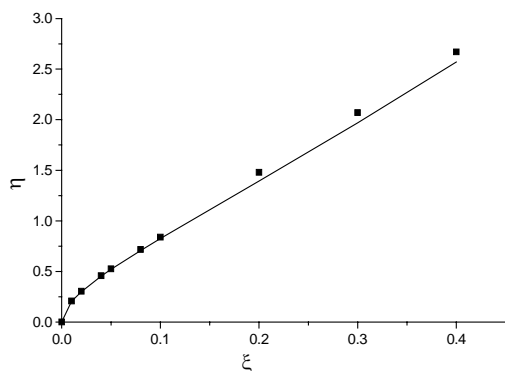


图 4 当 $\lambda = 0.3$ 时 η 随 ξ 的变化

Fig.4 Dependence of η on ξ when $\lambda = 0.3$

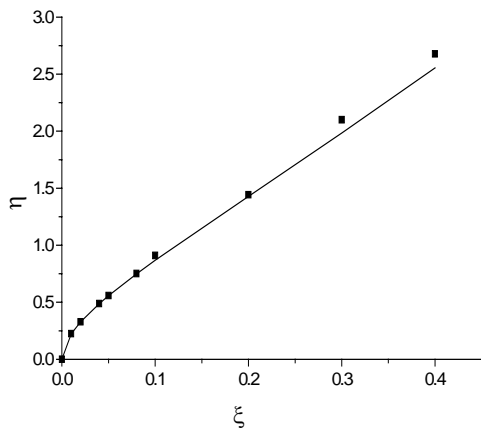


图 5 当 $\lambda = 0.5$ 时 η 随 ξ 的变化

Fig.5 Dependence of η on ξ when $\lambda = 0.5$

根据式(31)可以确定在无限大板中裂纹面受两集中剪力作用下的承载力,图 3~图 6 表示本文数值结果和 Achenbach 和 Dunayevsky^[2]结果比较,其中黑点为 Achenbach 和 Dunayevsky 结果^[2]。从图 3~图 6 可以知道本文结果和 Achenbach 和 Dunayevsky^[2]结果较吻合,因而说明本文结果的正确性。

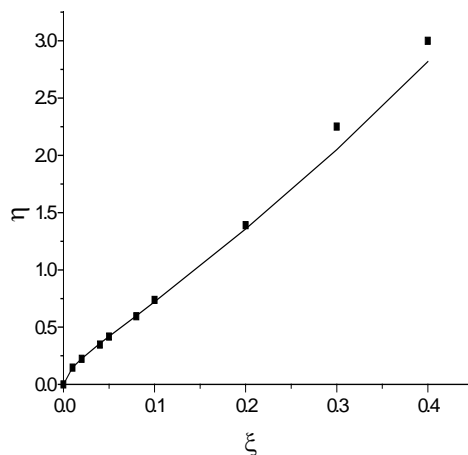


图 6 当 $\lambda = 0.8$ 时 η 随 ξ 的变化

Fig.6 Dependence of η on ξ when $\lambda = 0.8$

2 结果的讨论

(1) 根据式(28)或式(31)可以确定裂纹线附近塑性区的长度,即 r_0 或 ξ 。

(2) 式(29)和式(30)可以确定塑性区应力场的积分常数 B 。由于积分常数 B 的存在,裂纹尖端一般不会出现应力奇异现象。但是当 $B \leq 0$ 时,裂纹尖端有可能出现应力奇异现象。这种奇异性和传统意义或小范围屈服意义下的奇异性有本质的区别,它不一定出现在裂纹尖端处,这必然是裂纹开始起裂扩展或失稳扩展。此外, $B < 0$, 在裂纹尖端附近应变出现负值,是实际不可能的情况。因此,本文要求 $B \geq 0$, 并且 $B = 0$ 是其极值情况。

(3) 根据式(29)和式(30)可以确定裂纹线附近弹塑性边界上的单位法向矢量,进而可以预测整个塑性区的变化。

3 结论

(1) 本文利用裂纹线场方法对理想弹塑性材料无限大板受两集中剪力问题进行了弹塑性分析。求得了裂纹线附近的弹塑性解析解、裂纹线上的塑性区长度随荷载的变化规律。

(2) 本文的分析完全放弃了小范围屈服理论的所有近似假设,并且不再附加任何其他假设,因此

本文求解的裂纹面受两集中剪力作用下的裂纹线附近的塑性区的应力场和弹性区的应力场在裂纹线附近足够精确。

参考文献:

- [1] Achenbach J D, Li Z L. Plane stress crack line fields for crack growth in an elastic-perfectly plastic material [J]. *Engineering Fracture Mechanics*, 1980, 20(3): 534~544.
- [2] Achenbach J D, Dunayevsky V. Crack growth under plane stress condition in an elastic-perfectly plastic material [J]. *Journal of the Mechanics and Physics of Solids*, 1984, 32(2): 89~100.
- [3] Guo Q, Li K. The dimension of plastic zone ahead of a mode plane stress crack [J]. *Engineering Fracture Mechanics*, 1988, 29(1): 107~112.
- [4] Yi Z. The new and analytical solutions for mode cracks in an elastic-perfectly material [J]. *Engineering Fracture Mechanics*, 1992, 42(5): 833~840.
- [5] Yi Z. Precise solutions of elastic-plastic crack line fields for crack loaded by antiplane point forces [J]. *Engineering Fracture Mechanics*, 1997, 57(1): 75~83.
- [6] 吴承平, 王成. 裂纹面任意点受反平面集中力时裂纹线的弹塑性分析[J]. *应用数学和力学*, 1996, 17(12): 1059~1064.
- [7] Wu Chengping, Wang Cheng. Near crack line elastic-plastic analysis for a crack loaded by antiplane point forces [J]. *Applied Mathematic and Mechanics*, 1996, 17(12): 1059~1064. (in Chinese)
- [7] Zhou Xiaoping, Wang Jianhua. Near crack line elastic-plastic analysis for an infinite plate loaded by a pair of point shear forces [J]. *Journal of Shanghai Jiaotong University*, 2003, E-8 (2): 115~117.
- [8] Wang Jianhua, Zhou Xiaoping. Near crack line elastic-plastic analysis for an infinite plate loaded by two pairs of point tensile forces [J]. *Mechanics Research Communications*, 2004, 31(4): 415~420.
- [9] Zhou Xiaoping, Wang Jianhua. Study on the coalescence mechanism of splitting failure of crack-weakened rock subjected to compressive loads [J]. *Mechanics Research Communications*, 2005, 32(2): 161~171.
- [10] 周小平, 张永兴, 朱可善. 偏心裂纹面受一对集中拉力作用下的弹塑性理论解[J]. *固体力学学报*, 2005, 26(3): 267~272.
- Zhou Xiaoping, Zhang Yongxing, Zhu Keshan. Elastic-plastic analytical solution for an eccentric crack loaded by a pair of tensile point forces [J]. *Acta Mechanica Solida Sinica*, 2005, 26(3): 267~272. (in Chinese)

(上接第 52 页)

参考文献:

- [1] Baruch M. Optimization procedure to correct stiffness and flexibility matrices using vibration test [J]. *AIAA Journal*, 1978, 6(11): 1208~1210.
- [2] Law S S, Shi Z Y, Zhang L M. Structural damage detection from incomplete and noisy modal test data [J]. *J. of Engr. Mech, ASCE*, 1998, 124(11): 1280~1288.
- [3] Ren W X, Roeck G D. Structural damage identification using modal data I: Simulation verification [J]. *J. of Struc. Engr, ASCE*, 2002, 128(1): 87~95.
- [4] Doebling S W, Farrar C R, Prime M B, Shevitz D W. Damage identification and health monitoring of structural and mechanical systems from changes in their vibration characteristics: A Literature Review [R]. *Research Rep. Los Alamos National Lab., LA-13070-MS, UC-900, Los Alamos, NM, May, 1996.*
- [5] Stubbs N, Skikorsky C, Park S, Choi S, Bolton R. A methodology to nondestructively evaluate the structural properties of bridges [A]. *Proc. IMAC17, the 17th Int. Modal Analysis Conf, Kissim-mee, Fla, 1999, 1260~1268.*
- [6] Kim J T, Stubbs N. Model-uncertainty impact and damage detection accuracy in plate girder [J]. *J. Structural Engineering, ASCE*, 1995, 121: 1409~1417.
- [7] 张蓓, 殷学纲. 结构损伤的概率诊断[J]. *重庆大学学报(自然科学版)*, 2000, 23(2): 60~63.
- Zhang Bei, Yin Xuegang. Probability diagnosis of structural failure [J]. *Journal of Chongqing University*, 2000, 23(2): 60~63. (in Chinese)
- [8] Papadopoulos L, Garcia E. Structural damage identification: A probabilistic approach [J]. *AIAA J.*, 1998, 2137~2145.
- [9] Yong Xia, Hong Hao, James M W Brownjohn. Damage identification of structures with uncertain frequency and mode shape data [J]. *Earthquake Engineering and Structural Dynamics*, 2002, 31: 1053~1066.
- [10] 黄斌. 一种新的谱随机有限元方法[J]. *武汉理工大学学报*, 2004, 26(5): 42~45.
- Huang Bin. A new spectral random finite element method [J]. *Journal of Wuhan Univ. of Tech.*, 2004, 26(5): 42~45. (in Chinese)